

# Frenet-Serret dynamics

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**Abstract.** We consider the motion of a particle described by an action that is a functional of the Frenet-Serret [FS] curvatures associated with the embedding of its worldline in Minkowski space. We develop a theory of deformations tailored to the FS frame. Both the Euler-Lagrange equations and the physical invariants of the motion associated with the Poincaré symmetry of Minkowski space, the mass and the spin of the particle, are expressed in a simple way in terms of these curvatures. The simplest non-trivial model of this form, with the Lagrangian depending on the first FS (or geodesic) curvature, is integrable. We show how this integrability can be deduced from the Poincaré invariants of the motion. We go on to explore the structure of these invariants in higher-order models. In particular, the integrability of the model described by a Lagrangian that is a function of the second FS curvature (or torsion) is established in a three dimensional ambient spacetime.

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## 1. Introduction

The simplest action describing the motion of a particle is proportional to the proper time along the trajectory of the particle in spacetime, or worldline. When there are no external fields, this is the unique Poincaré invariant action that can be constructed using only the velocity of the particle.

In this paper, we examine a hierarchy of geometrical models describing particle motion that involve successively higher derivatives of the embedding functions of the worldline [1, 2, 3, 4, 5, 6, 7, 8, 9]. Such models arise as a description of an isolated object when its internal structure is not resolved. Indeed, the effective bosonic theory describing a supersymmetric particle when fermionic degrees of freedom have been integrated out can itself be cast as a non-local theory of this form [10]. We will focus, however, not so much on any specific model, but on the relationship between the order of a given local geometrical action and the dimension of the background geometry. While this point is trivial for a free particle, it is not so for higher order models.

Let  $N + 1$  be the dimension of our spacetime. Suppose also that the trajectory is sufficiently smooth so that the Frenet-Serret [FS] frame which is adapted to it [11, 12] is defined, namely, the velocity  $V$  along the trajectory is differentiable  $N$  times. We can then associate a curvature  $\kappa_n$  with each of the first  $N$  derivatives of  $V$ :  $\kappa_1$  is the acceleration of the particle;  $\kappa_2$  describes the torsion of its trajectory and so on. Once the FS curvatures have been specified the course of the particle can be reconstructed

up to a Poincaré rigid motion (as in [13], Corollary 7.4). In this way the frame provides a complete kinematical description of the motion. Every geometrical scalar associated with the trajectory can be constructed using only the FS curvatures and their derivatives. We thus abandon the embedding functions as our primary variables; their derivatives replaced by geometrically meaningful objects and cast the action as a functional of the FS curvatures,

$$S = \int d\tau L(\kappa_1, \kappa_2, \dots, \kappa'_1, \dots),$$

where  $\tau$  is the proper time along the trajectory of the particle and the prime denotes a derivative with respect to  $\tau$ .

In its original Euclidean incarnation, the Lagrangian  $L(\kappa_1)$  can be traced back to the beginning of the calculus of variations in Euler's description of space curves (see *e.g.* [14]). The model is exactly solvable, a fact not at all obvious if the action is expressed in terms of its original embedding variables. By adapting the differential geometry of Pfaffian forms to the FS frame, however, Dereli *et al.* were able to show that the relativistic particle model was integrable by expressing  $\kappa_1$  as a quadrature [11]. An alternative approach without Pfaffian forms was developed recently by Nesterenko *et al.* [12].

Beyond the first curvature, there has been very little progress made to date. This is understandable: the equations of motion are an intractable muddle when expressed directly in terms of the embedding functions; the Hamiltonian formalism based on them, preferred by physicists with their sight set on quantization, even worse. To some extent, we are able to overcome this obstacle by developing a theory of worldline deformations which is tailored to the FS frame: the deformation in the worldline embedding functions is decomposed with respect to this frame; the infinitesimal change induced in the curvatures under this deformation is cast as a linear differential operator acting on these projections. The expressions themselves turn out to be far simpler than the corresponding expressions for the deformation of the relevant order of derivatives of the embedding functions. The FS frame projections also provide a natural set of independent deformations in the variational principle; the projections of the Euler-Lagrange equations along these directions assume a strikingly simple form.

The early work on  $L(\kappa_1)$  demonstrated very clearly the natural division of the dynamical problem into two parts: in the first part two first integrals of the equations of motion in terms of the curvature of the worldline and its first derivative are identified and combined to provide a quadrature; in the second part the FS equations are integrated for the position vector of the worldline of the particle for the given  $\kappa_1$ . In higher order systems, it is not easy to identify first integrals. We are guided, however, by Nesterenko *et al.* who recognized in the context of  $L(\kappa_1)$ , that these first integrals are none other than the Casimir invariants of the motion which are a consequence of the Poincaré symmetry of Minkowski space, the mass  $M^2$  and the spin  $S^2$ , of the particle associated with the conserved linear and angular momenta respectively.

We will employ Noether's theorem to determine the Poincaré invariants for any local Lagrangian constructed from the FS curvatures. Thus, there always exist two non-trivial first integrals of the equations of motion. Using the deformation theory we have developed, these invariants can be cast in a simple as well as instructive way in terms of the FS curvatures.

In the case of  $L(\kappa_1)$ , the conservation of angular momentum identifies  $\kappa_2$  as a function of  $\kappa_1$ . This permits us to cast the second integral of the motion as a

quadrature,  $\kappa_1'^2 + V(\kappa_1; S^2, M^2) = 0$ , which mimics the radial motion of a fictitious non-relativistic particle in a central potential on the normal plane.

In general, the system of equations will not be completely integrable: the first integrals do not provide a quadrature. This appears to be the case for  $L(\kappa_1, \kappa_2)$ . Remarkably, however, we show that  $L(\kappa_2)$ , like the lower order  $L(\kappa_1)$ , is completely integrable when  $N = 2$  (the analogous result in three-dimensional Euclidean space will be considered elsewhere). This time, however, the integrals of motion are entangled in a rather less trivial way. Nonetheless, the first integral of the equations of motion can be cast as a quadrature for  $\kappa_2$ ,  $\kappa_2'^2 + V(\kappa_2; S^2, M^2) = 0$ , and  $\kappa_1$  as function of  $(\kappa_2, S^2, M^2)$ , thus once again permitting one to determine the curve up to a rigid motion. The equations of motion never need to be integrated explicitly.

The paper is organized as follows: In Sect. 2. we summarize the relevant worldline geometry. In Sect. 3. we develop the theory of deformations of an arbitrary worldline. In Sect. 4., we examine general features of relativistic particle described by any (local) Lagrangian of the form  $L(\kappa_1, \kappa_2, \dots, \kappa'_1, \dots)$ . In Sect. 5 we specialize to a Lagrangian,  $L = L(\kappa_1)$ . In Sect. 6 we extend our treatment to models which depend on the second curvature,  $L = L(\kappa_2)$ .

## 2. Worldline geometry

In this section we consider the geometry of embedded timelike curves from two different points of view. First we use a language which is the one-dimensional version of the geometry of higher dimensional embedded surfaces in an arbitrary parametrization. Next we consider the description in terms of the FS basis for the curve, which exploits the existence of a preferred parameter, proper time. Finally we show how these two descriptions are related. To our knowledge, this relationship has not been explored before. We demonstrate that, whereas the higher order FS curvatures assume an impenetrable form when expressed in terms of derivatives of the embedding functions, they assume a remarkably simple form in terms of the unit  $k^i$  and their derivatives. Indeed, the top FS curvature is a simple polynomial in these variables.

We consider a relativistic particle whose trajectory is described by the worldline,  $x^\mu = X^\mu(\xi)$ , embedded in Minkowski spacetime  $\{M, \eta_{\mu\nu}\}$  of dimension  $N + 1$  (we use a signature with only one minus sign;  $\mu, \nu, \dots = 0, 1, \dots, N$ ).  $\xi$  is an arbitrary parameter. The vector tangent to the curve is  $\dot{X}^\mu$ , where the overdot represents a derivative with respect to the parameter  $\xi$ . The one-dimensional induced metric on the curve is  $\gamma = \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} = \dot{X} \cdot \dot{X}$ . The infinitesimal proper time elapsed along the trajectory is then given by  $d\tau = \sqrt{-\gamma} d\xi$ . If one uses proper time as the parameter (taking advantage of the fact that the intrinsic geometry of the curve is trivial) then  $\gamma = -1$ .

In our first description of the geometry, the normal frame is not fixed: an arbitrary set of vectors normal to the curve,  $n^\mu_i$ , can be defined implicitly by  $n_i \cdot \dot{X} = 0$ , and normalized  $n_i \cdot n_j = \delta_{ij}$  ( $i, j \dots = 1, 2, \dots, N$ ). Normal indices are raised and lowered with the Kronecker delta. The one-dimensional analogue of the Gauss-Weingarten equations is given by

$$\ddot{X} = \Gamma \dot{X} - K^i n_i, \quad \dot{n}_i = k_i \dot{X} + \omega_i^j n_j. \quad (1)$$

Here we have introduced the one-dimensional affine connection  $\Gamma = \gamma^{-1} \dot{X} \cdot \ddot{X} = (1/2) \gamma^{-1} \dot{\gamma}$ . We define the covariant derivative under worldline reparametrizations  $\nabla = (d/d\xi) - \Gamma$ . The connection  $\Gamma$  vanishes in the parametrization by proper time. The

extrinsic curvature along the  $i$ -th normal is  $K_i = -n_i \cdot \ddot{X}$ , and the point-like analogue of the mean extrinsic curvature is the scalar under reparametrizations  $k_i = \gamma^{-1} K_i$ . The  $K_i$  are genuine curvatures involving second derivatives of the embedding, as distinct from the Frenet-Serret curvatures which we discuss below.

The arbitrariness in the choice of frame does however come at a price - an additional one-dimensional connection needs to be introduced to maintain covariance under normal rotations. The extrinsic twist, or normal form,  $\omega_{ij} = n_j \cdot \dot{n}_i$  is the connection associated with the freedom of rotations of the normals. We use it to define the covariant derivative under rotation of the normals  $\hat{\nabla} = \nabla - \omega$ . We will also use the symbol  $\hat{D}$  to denote the covariant derivative  $\hat{\nabla}$  using proper time as the parameter.

The existence of a natural parameter, proper time, leads to an alternative description, unique to curves, which is the generalization to higher ambient space dimensions of the classical FS formalism for space curves (see *e.g.* [15], [13]). In this approach, one introduces an orthonormal basis  $\{X', \eta_i\}$ , where a prime denotes derivative with respect to proper time. The FS equations for a curve in  $N + 1$  dimensions are

$$\begin{aligned} X'' &= \kappa_1 \eta_1, \\ \eta_1' &= \kappa_1 X' - \kappa_2 \eta_2, \\ \eta_2' &= \kappa_2 \eta_1 - \kappa_3 \eta_3, \\ &\dots \dots \dots \\ \eta_{N-1}' &= \kappa_{N-1} \eta_{N-2} - \kappa_N \eta_N, \\ \eta_N' &= \kappa_N \eta_{N-1}. \end{aligned} \tag{2}$$

where  $\kappa_i$  represents the  $i^{\text{th}}$  FS curvature. In an ambient space of dimension  $N + 1$ , there are at most  $N$  FS curvatures.  $\kappa_1$  is the geodesic curvature. Implicit in this definition is that the embedding functions  $X^\mu$  are  $N + 1$  times differentiable, and that the  $\kappa_i$  do not vanish. If they vanish only at isolated points however the FS basis can still be defined consistently (see [13], Ch. 7.)

There are some important results of the geometry of curves that we will use extensively in this paper. First, the fundamental theorem for curves says that for a curve embedded in an  $N + 1$ -dimensional space its  $N$  FS curvatures determine the curve, up to rigid motions. The actual trajectory can always be obtained from the curvatures by quadratures. This implies that the curvatures can be used as a set of natural auxiliary variables for the description of curves. In addition, if the  $i$ -th FS curvature vanishes, so do all the higher ones and, moreover, the curve will lie in an  $i$ -dimensional subspace. For example, if  $\kappa_2 = 0$  then the motion lies on a Minkowski plane, if  $\kappa_3 = 0$  the motion is in  $2 + 1$  dimensions. (For the Euclidean case see [13], Th. 7.5.) Let us also remark that the top FS curvature  $\kappa_N$  has a special status. It is possible to express its content in a way which depends on the orientation of the ambient space. This *signed* curvature will be denoted with an overbar,  $\bar{\kappa}_N$ , so that (see below for an explicit definition of  $\bar{\kappa}_N$ , valid for any  $N$ )

$$\bar{\kappa}_N^2 = \kappa_N^2. \tag{3}$$

Let us turn our attention now to the problem of relating these two descriptions of the geometry of curves. We will express the FS curvatures in terms of  $k^i$  and its derivatives. In addition, we obtain an explicit expression for the curvature dependent rotation matrix taking an arbitrary normal basis  $n_i$  into FS normal form  $\eta_i$ :

$$\eta_i = R_i^j n_j. \tag{4}$$

We begin by parametrizing Eq.(1) by proper time:

$$X'' = k^i n_i, \quad \tilde{D}n_i = k_i X', \quad (5)$$

where the symbol  $\tilde{D}$  was defined below Eq.(1). Now, comparison of the first with the first of (2) gives immediately  $\kappa_1 \eta_1 = k^i n_i$ . It follows that the first FS curvature is simply the modulus of  $k^i$ ,  $\kappa_1 = k = \sqrt{k^i k_i}$ . We have also that  $\eta_1 = \hat{k}^i n_i$ , where we have introduced the unit vector  $\hat{k}^i = k^i/k$ , so that we find  $R_1^i = \hat{k}^i$ . For the next order, we take a derivative of  $\eta_1$ , and using (5), we obtain  $\eta_1' = (\tilde{D}\hat{k}^i)n_i + \kappa_1 X'$ . Comparison with the second FS equation gives

$$\kappa_2 = \sqrt{\tilde{D}\hat{k}^i \tilde{D}\hat{k}_i}, \quad (6)$$

and  $\eta_2 = -\kappa_2^{-1}(\tilde{D}\hat{k}^i)n_i$ , so that  $R_2^i = -\kappa_2^{-1}\tilde{D}\hat{k}^i$ . Note that  $\eta_1 \cdot \eta_2 = 0$  follows from the unit vector fact  $\hat{k}_i \tilde{D}\hat{k}^i = 0$ . The interesting non-obvious fact is that the second FS curvature is simply the modulus of  $\tilde{D}\hat{k}^i$ . We can continue in the same fashion for the higher FS curvatures. One finds that  $\kappa_n$  is proportional to the modulus of the projection of  $\tilde{D}^{n-1}\hat{k}^i$  orthogonal to all lower derivatives of  $\hat{k}^i$ . What becomes clear is that the  $N-1$  derivatives of  $k^i$ , together with  $k^i$  itself provide a basis for the normal directions. This implies that one can construct a set of normal forms that encode completely the information contained in the higher FS curvatures:

$$T_{i_1 \dots i_{N-r}}^{(N)} = \frac{1}{\sqrt{(N-r)!}} \frac{1}{k^{r-1}} \varepsilon_{i_1 \dots i_{N-r} j_1 \dots j_r} k^{j_1} (\tilde{D}k^{j_2}) \dots (\tilde{D}^{r-1}k^{j_r}), \quad (7)$$

where,  $r = 1, \dots, N$ . Here  $\varepsilon_{i_1 \dots i_N} = \varepsilon_{\alpha_1 \dots \alpha_N \mu} n^{\alpha_1}_{i_1} \dots n^{\alpha_N}_{i_N} X^{\mu'}$  is the normal Levi-Civita density, with  $\varepsilon_{\alpha_1 \dots \alpha_N \mu}$  the Minkowski background Levi-Civita density. Note that we have the alternative expression in terms of the unit  $\hat{k}^i$  and its derivatives,

$$T_{i_1 \dots i_{N-r}}^{(N)} = \frac{1}{\sqrt{(N-r)!}} k \varepsilon_{i_1 \dots i_{N-r} j_1 \dots j_r} \hat{k}^{j_1} \tilde{D}\hat{k}^{j_2} \dots (\tilde{D}^{r-1}\hat{k}^{j_r}). \quad (8)$$

For example, for a space curve we have  $T_i^{(2)} = k \varepsilon_{ij} \hat{k}^j$ , and  $T^{(2)} = k \varepsilon_{ij} \hat{k}^i \tilde{D}\hat{k}^j$ , whereas for a curve in four dimensions,

$$T_{ij}^{(3)} = \frac{k}{\sqrt{2}} \varepsilon_{ijl} \hat{k}^l, \quad T_i^{(3)} = k \varepsilon_{ijl} \hat{k}^j \tilde{D}\hat{k}^l, \quad T^{(3)} = k \varepsilon_{ijl} \hat{k}^i (\tilde{D}\hat{k}^j) (\tilde{D}^2\hat{k}^l). \quad (9)$$

These normal forms contain the same information as the FS curvatures, in the sense that the following relation holds,

$$T_{i_1 \dots i_{N-r}}^{(N)} T^{(N)i_1 \dots i_{N-r}} = \kappa_1^2 \kappa_2^2 \dots \kappa_r^2. \quad (10)$$

For the signed top FS curvature we find the relation

$$T^{(N)} = \kappa_1 \kappa_2 \dots \kappa_{N-1} \bar{\kappa}_N, \quad (11)$$

which provides a simple definition for  $\bar{\kappa}_N$  valid for any  $N$  that satisfies (3). The expression for  $\bar{\kappa}_N$  directly in terms of the embedding functions is much more complicated. We note, in particular, that the (signed) torsion of the worldline (a spacetime pseudoscalar) is defined with respect to the embedding functions by

$$\bar{\kappa}_2 = \frac{\varepsilon_{\mu\nu\rho} \dot{X}^\mu \ddot{X}^\nu \ddot{X}^\rho}{\dot{X}^2 \ddot{X}^2 - (\dot{X} \cdot \ddot{X})^2}, \quad (12)$$

whereas in terms of  $\hat{k}^i$  and its derivative it assumes the remarkably simple polynomial form,  $\bar{\kappa}_2 = \varepsilon_{ij} \hat{k}^i \tilde{D}\hat{k}^j$ .

An alternative approach is the following: let us introduce the objects  $\Pi_{(n+1)}^{ij}$ , defined recursively by ( $n \leq N$ )

$$\Pi_{(n+1)}^{ij} = \Pi_{(n)}^{ij} - \frac{1}{\kappa_n^2 \cdots \kappa_1^2} \Pi_{(n)}^i{}_k \Pi_{(n)}^{j l} (\tilde{D}^{n-1} k^k) (\tilde{D}^{n-1} k^l), \quad (13)$$

with  $\Pi_{(1)}^{ij} = \delta^{ij}$ . The lowest cases are

$$\Pi_{(2)}^i{}_j = \delta^i{}_j - \hat{k}^i \hat{k}_j, \quad \Pi_{(3)}^i{}_j = \delta^i{}_j - \hat{k}^i \hat{k}_j - \frac{1}{\kappa_2^2} (\tilde{D} \hat{k}^i) (\tilde{D} \hat{k}_j).$$

These objects are projection matrices,

$$\Pi_{(n)}^i{}_j \Pi_{(n)}^{j k} = \Pi_{(n)}^i{}_k, \quad \Pi_{(n)}^i{}_i = N - n + 1, \quad (14)$$

which satisfy

$$\Pi_{(n)}^i{}_j \Pi_{(m)}^{j k} = \Pi_{(n)}^i{}_k, \quad \Pi_{(n)}^i{}_j \tilde{D}^{m-1} k^j = 0 \quad (n > m) \quad (15)$$

In general, we have the following relation with the normal forms introduced earlier,

$$\Pi_{(n)}^i{}_j = \frac{(N - n + 1)}{\kappa_1^2 \cdots \kappa_{n-1}^2} T^{(N) i i_1 \cdots i_{N-n}} T^{(N)}_{j i_1 \cdots i_{N-n}}. \quad (16)$$

With the help of these projectors, we find that the FS curvatures are defined recursively by

$$\kappa_n = \frac{1}{\kappa_{n-1} \kappa_{n-2} \cdots \kappa_1} \sqrt{(\tilde{D}^{n-1} k_i) \Pi_{(n)}^{ij} (\tilde{D}^{n-1} k_j)}. \quad (17)$$

Using (16), it is easy to show that this expression agrees with what we have found earlier, (10). For  $\kappa_2$  it reduces to the expression (6). The next FS curvature can be written as

$$\kappa_3 = \frac{1}{\kappa_2} \sqrt{(\tilde{D}^2 \hat{k}^i) (\tilde{D}^2 \hat{k}_i) - (\kappa_2')^2 - \kappa_2^4}. \quad (18)$$

We emphasize that this is a considerable improvement on the corresponding expression directly in terms of derivatives of the embedding functions.

We also obtain the sought for general expression (up to a sign) for the rotation matrices defined by (4), as

$$R_i^j = \frac{1}{\kappa_i \kappa_{i-1} \cdots \kappa_1} \Pi_{(i)}^{j l} \tilde{D}^{i-1} k_l. \quad (19)$$

To conclude this section we note that the expressions we have derived generalize in a straightforward way to the case of an arbitrary curved background spacetime  $\{M, g_{\mu\nu}\}$ . We would have  $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$  and derivation by proper time becomes the covariant derivative along the tangent vector to the worldline. In particular, this changes the definition of  $k^i$ , which depends now on the background covariant derivative.

### 3. Worldline deformations

In this section we analyze the change in the geometry of the worldline due to an infinitesimal deformation  $X^\mu(\xi) \rightarrow X^\mu(\xi) + \delta X^\mu(\xi)$ . Let us first decompose the deformation in its tangential and normal parts with respect to the basis  $\{X', n_i\}$ ,

$$\delta X = \Phi X' + \Phi^i n_i. \quad (20)$$

Tangential deformations are reparametrizations of the worldline. All that we will need below is the tangential variation of the infinitesimal proper time,

$$\delta_{||} d\tau = d\tau \Phi'. \quad (21)$$

Now consider the normal part of the deformation. Its effect on the geometry of a normal deformation can be derived directly, or obtained by considering the point-like limit of the expressions obtained in [16] for relativistic extended objects. For the normal deformation of the infinitesimal proper time we have

$$\delta_{\perp} d\tau = d\tau k_i \Phi^i. \quad (22)$$

For the normal deformation of the extrinsic curvature along the  $i$ -th normal we find

$$\tilde{\delta}_{\perp} K^i = -\tilde{\nabla}^2 \Phi^i + k^i K_j \Phi^j.$$

The symbol  $\tilde{\delta}$  denotes the normal rotation covariant deformation operator defined in [16]. With these expressions we obtain that the normal variation of the mean curvature along the  $i$ -th normal is

$$\tilde{\delta}_{\perp} k^i = -\gamma^{-1} \tilde{\nabla}^2 \Phi^i - k^i k_j \Phi^j = \tilde{D}^2 \Phi^i - k^i k_j \Phi^j, \quad (23)$$

so that, using  $\kappa_1 = k$ , this gives

$$\delta_{\perp} \kappa_1 = \hat{k}_i \left( \tilde{D}^2 \Phi^i - \kappa_1^2 \Phi^i \right). \quad (24)$$

In order to evaluate the variation of derivatives of the curvature we need also the normal variation of the extrinsic twist,

$$\tilde{\delta}_{\perp} \omega_{ij} = k_j \tilde{\nabla} \Phi_i - k_i \tilde{\nabla} \Phi_j.$$

It follows that, for example,

$$\tilde{\delta}_{\perp} \left( \tilde{\nabla} k^i \right) = -\gamma^{-1} \tilde{\nabla}^3 \Phi^i - k^2 \tilde{\nabla} \Phi^i - \Phi^j \tilde{\nabla} (k^i k_j). \quad (25)$$

These expressions are sufficient to obtain the variation of any geometrical invariant for the worldline. However, as one considers higher FS curvatures, the resulting expressions become increasingly unmanageable. For this reason it becomes desirable to develop an alternative approach where the normal deformation is expanded with respect to the FS basis as (we are suspending for the remainder of this section the summation convention)

$$\delta_{\perp} X = \sum_i \Psi_i \eta_i. \quad (26)$$

Note that the normal components of the deformation are related by a rotation,  $\Phi^i = \sum_j R^i_j \Psi^j$ . Since  $k_i = \kappa_1 n_i \cdot \eta_1$ , we have that the normal variation of the infinitesimal proper time takes the form

$$\delta_{\perp} d\tau = d\tau \kappa_1 \Psi_1. \quad (27)$$

It follows that the commutator between normal variation and derivation by proper time is

$$\left[ \delta_{\perp}, \frac{d}{d\tau} \right] = -\kappa_1 \Psi_1, \quad (28)$$

so that in particular for the deformation of the tangent vector we have

$$\delta_{\perp} X' = -\kappa_1 \Psi_1 X' + \left( \sum_i \Psi^i \eta_i \right)'.$$

Using the FS equations and rearranging the sum, this expression reduces to

$$\delta_{\perp} X' = \sum_i \left[ \Psi^{i'} \eta_i + \kappa_{i+1} \Psi^{i+1} \eta_i - \kappa_{i+1} \Psi^i \eta_{i+1} \right]. \quad (29)$$

The projections of  $\delta_{\perp} X'$  along the FS normals are then

$$\eta^i \cdot \delta_{\perp} X' = \Psi^{i'} + \kappa_{i+1} \Psi^{i+1} - \kappa_i \Psi^{i-1}, \quad (30)$$

where  $\Psi^i = 0$  if  $i < 1$  or  $i > N$ . The corresponding projections of the variation of the acceleration  $X''$  is given by

$$\begin{aligned} \eta^i \cdot \delta_{\perp} X'' &= \kappa_{i+2} \kappa_{i+1} \Psi^{i+2} + 2\kappa_{i+1} \Psi^{i+1'} + \kappa_{i+1}' \Psi^{i+1} + \Psi^{i''} - (\kappa_i^2 + \kappa_{i+1}^2) \Psi^i \\ &\quad - 2\kappa_i \Psi^{i-1'} - \kappa_i' \Psi^{i-1} + \kappa_i \kappa_{i-1} \Psi^{i-2}, \end{aligned} \quad (31)$$

where again  $\Psi_i = 0$  if  $i < 1$  or  $i > N$ . Eq.(31) follows from

$$\eta^i \cdot \delta_{\perp} X'' = (\eta^i \cdot \delta_{\perp} X')' - \eta^{i'} \cdot \delta_{\perp} X' - \kappa_1 \Psi_1 (\eta^i \cdot X''),$$

the FS equations and (30). Remarkably, this is all we need to evaluate the variations of the FS curvatures. We will do this explicitly for  $\kappa_1$ ,  $\kappa_2$  and  $\kappa_3$  (the latter only in a four-dimensional background). The generalization to higher curvatures can be worked out along the same lines.

Taking a variation of the first of (2) we have

$$\delta_{\perp} X'' = (\delta_{\perp} \kappa_1) \eta_1 + \kappa_1 \delta_{\perp} \eta_1. \quad (32)$$

Using the orthogonality of  $\eta_1$  and  $\delta_{\perp} \eta_1$  we obtain  $\delta_{\perp} \kappa_1 = \eta_1 \cdot \delta_{\perp} X''$ , and specializing (31) to  $i = 1$ , we find

$$\delta_{\perp} \kappa_1 = \kappa_3 \kappa_2 \Psi_3 + 2\kappa_2 \Psi_2' + \kappa_2' \Psi_2 + \Psi_1'' - (\kappa_1^2 + \kappa_2^2) \Psi_1. \quad (33)$$

This expression should be compared with (24), using  $\Phi^i = \sum R_j^i \Psi^j$ . Note that the normal variation of  $\kappa_1$  involves at most three normal directions.

To evaluate the variation of the second FS curvature, we take a variation of the second of (2) and dot with  $\eta_2$ . We obtain

$$\delta_{\perp} \kappa_2 = \kappa_1 \eta_2 \cdot \delta_{\perp} X' - \eta_2 \cdot \delta_{\perp} \eta_1'.$$

We can rewrite the second term on the right hand side as

$$\begin{aligned} \eta_2 \cdot \delta_{\perp} \eta_1' &= -\kappa_1 \Psi_1 (\eta_2 \cdot \eta_1') + \eta_2 \cdot (\delta_{\perp} \eta_1)' \\ &= \kappa_1 \kappa_2 \Psi_1 + (\eta_2 \cdot \delta_{\perp} \eta_1)' - \eta_2' \cdot \delta_{\perp} \eta_1 \\ &= \kappa_1 \kappa_2 \Psi_1 + \left[ \frac{1}{\kappa_1} \eta_2 \cdot \delta_{\perp} X'' \right]' - \frac{\kappa_2}{\kappa_1} \eta_1 \cdot \delta_{\perp} X'' + \frac{\kappa_3}{\kappa_1} \eta_3 \cdot \delta_{\perp} X'', \end{aligned}$$



where we have used the second FS equation in the second line and the third in the last line.  $\delta_\perp \kappa_2$  is now expressed in terms of known quantities. A little algebra gives

$$\begin{aligned} \delta_\perp \kappa_2 &= \kappa_1(\Psi_2' - 2\kappa_2\Psi_1) \\ &- \left\{ \frac{1}{\kappa_1} \left[ \Psi_2'' - 2\kappa_2\Psi_1' - \kappa_2'\Psi_1 - (\kappa_2^2 + \kappa_3^2)\Psi_2 + 2\kappa_3\Psi_3' + \kappa_3'\Psi_3 + \kappa_3\kappa_4\Psi_4 \right] \right\}' \\ &- \frac{\kappa_3}{\kappa_1} \left[ \kappa_2\kappa_3\Psi_1 - 2\kappa_3\Psi_2' - \kappa_3'\Psi_2 + \Psi_3'' - (\kappa_1^2 + \kappa_3^2 + \kappa_4^2)\Psi_3 \right. \\ &\quad \left. + 2\kappa_4\Psi_4' + \kappa_4'\Psi_4 + \kappa_4\kappa_5\Psi_5 \right]. \end{aligned} \quad (34)$$

Note that, whereas  $\delta_\perp \kappa_1$  involves three normal directions,  $\delta_\perp \kappa_2$  involves at most five. Let us remark that an important simplification occurs in this expression when we are in three dimensions and  $\kappa_2$  is the top FS curvature ( $\kappa_3 = \kappa_4 = \kappa_5 = 0$ ). The above expression reduces to

$$\delta_\perp \kappa_2 = \kappa_1(\Psi_2' - 2\kappa_2\Psi_1) - \left\{ \frac{1}{\kappa_1} \left[ \Psi_2'' - 2\kappa_2\Psi_1' - \kappa_2'\Psi_1 - \kappa_2^2\Psi_2 \right] \right\}'. \quad (35)$$

To complete the description of a particle moving in four dimensions one also requires an expression for the variation of the third FS curvature. Let  $N = 3$ , so that  $\kappa_4 = \kappa_5 = 0$ . We have

$$\delta_\perp \kappa_3 = \frac{\kappa_2}{\kappa_1} \eta_3 \cdot \delta_\perp X'' - \eta_3 \cdot \delta_\perp \eta_2'.$$

Using the FS equations the second term can be written as

$$\eta_3 \cdot \delta_\perp \eta_2' = \kappa_1\kappa_3\Psi_1 + (\eta_3 \cdot \delta_\perp \eta_2)'. \quad (36)$$

The second term in this expression can in turn be written as

$$\begin{aligned} \eta_3 \cdot \delta_\perp \eta_2 &= \frac{\kappa_1}{\kappa_2} \eta_3 \cdot \delta_\perp X' - \frac{1}{\kappa_2} \eta_3 \cdot \delta_\perp \eta_1' \\ &= \frac{\kappa_1}{\kappa_2} \eta_3 \cdot \delta_\perp X' - \frac{1}{\kappa_2} \left( \frac{1}{\kappa_1} \eta_3 \cdot \delta_\perp X'' \right)' + \frac{\kappa_3}{\kappa_2\kappa_1} \eta_2 \cdot \delta_\perp X''. \end{aligned}$$

Using (30) and (31), we obtain

$$\begin{aligned} \delta_\perp \kappa_3 &= -\kappa_1\kappa_3\Psi_1 + \frac{\kappa_2}{\kappa_1} \Psi_3'' - \frac{\kappa_2\kappa_3^2}{\kappa_1} \Psi_3 - 2\frac{\kappa_3\kappa_2}{\kappa_1} \Psi_2' - \frac{\kappa_2\kappa_3'}{\kappa_1} \Psi_2 + \frac{\kappa_3}{\kappa_1} \kappa_2^2 \Psi_1 \\ &- \left\{ \frac{\kappa_1}{\kappa_2} (\Psi_3' - \kappa_3\Psi_2) \right\}' + \left\{ \frac{1}{\kappa_2} \left[ \frac{1}{\kappa_1} (\Psi_3'' - \kappa_3^2\Psi_3 - 2\kappa_3\Psi_2' - \kappa_3'\Psi_2 + \kappa_3\kappa_2\Psi_1) \right] \right\}' \\ &- \left\{ \frac{\kappa_3}{\kappa_2\kappa_1} [\Psi_2'' - (\kappa_2^2 + \kappa_3^2)\Psi_2 + 2\kappa_3\Psi_3' + \kappa_3'\Psi_3 - 2\kappa_2\Psi_1' - \kappa_2'\Psi_1] \right\}'. \end{aligned} \quad (37)$$

Let us remark that in the case of a curved background, (31) is modified according to

$$\eta_i \cdot \delta_\perp X'' \rightarrow \eta_i \cdot \delta_\perp X''|_{flat} + R_{\mu\alpha\nu\beta} X^{\mu'} X^{\nu'} \eta^\alpha_j \eta^\beta_i \Psi^j, \quad (38)$$

where the first term is given by (31) and  $R_{\mu\alpha\nu\beta}$  denotes the background Riemann tensor. In order to obtain the normal variation of the FS curvatures in an arbitrary background one would have to make the appropriate changes in the expressions we have derived.

#### 4. Worldline invariants

The action functionals for relativistic particles we will consider satisfy various symmetry requirements. First of all, we consider only actions which are invariant under reparametrizations of the worldline of the form  $S = \int d\tau L$ , with  $L$  a scalar under reparametrizations, built out of the geometrical quantities that characterize the worldline, *i.e.* its curvatures. This assumption implies that the first variation of the action can always be written as

$$\delta S = \int d\tau E_i \Phi^i + \int d\tau Q', \quad (39)$$

where  $E_i$  denotes the Euler-Lagrange derivative, and  $Q$  is the Noether charge. This specific form follows from the fact that the tangential variation contributes only to the Noether charge. It is a simple consequence of the transformation properties of the action under reparametrizations. Indeed using (21) and the fact that  $\delta_{||}L = \Phi L'$ , the tangential part of the variation of the action is a total derivative

$$\delta_{||}S = \int d\tau (L\Phi)', \quad (40)$$

and does not contribute to the Euler-Lagrange part of the variation of the action.

The lowest order possibility is the relativistic massive particle, with a constant Lagrangian  $L_0 = -m$ , and an action proportional to the proper time elapsed along the trajectory. At higher order the Lagrangian will be a scalar constructed out of higher derivatives of the embedding functions  $L = L(X'', X''', \dots)$ . However writing higher order Lagrangians this way makes it increasingly difficult to keep track of the possible scalars one can construct at each order and to establish which ones are independent. Our approach is to construct the Lagrangian out of the  $k^i$  and its (covariant) derivatives  $L = L(k^i, \tilde{D}k^i, \tilde{D}^2k^i, \dots)$  or, equivalently, as we have established, as a function of the FS curvatures and their derivatives.

The lowest non-trivial geometrical models depend only on  $k^i$ . Invariance under rotations of the normals implies that the Lagrangian must depend on  $k^i$  only in the combination  $k$  or equivalently the first FS curvature,  $L = L(\kappa_1)$ . The next order allows a dependence on first derivatives of the  $k^i$ , so that  $L = L(k^i, \tilde{D}k^i)$ . Now what are the scalars we can construct at this order? A moment of thought reveals that the building blocks must be  $k, k', \tilde{D}k^i \tilde{D}k_i$ . There is one caveat: in the case of a three-dimensional ambient space, we have also the independent combination given by the pseudo-scalar  $\epsilon_{ij} k^i \tilde{D}k^j$ . In terms of the FS curvatures, at this order, we have  $L = L(\kappa_1, \kappa_1', \kappa_2)$ , and in the special case of a three-dimensional ambient space we have also the possibility  $L = L(\bar{\kappa}_2)$ . At the next order, we will have  $L = L(\kappa_1, \kappa_1', \kappa_1'', \kappa_2, \kappa_2', \kappa_3)$ .

#### 5. First-curvature models

In this section we consider a relativistic particle whose dynamics is determined by an action that depends at most on the first FS curvature of its worldline,  $L = L(\kappa_1)$ , where  $L$  is an arbitrary local function of its argument. We first derive the equations of motion and the Noether charges associated with the underlying Poincaré invariance for this class of models. The normal variation of the action is

$$\delta_{\perp}S = \int d\tau \left[ k_i L \Phi^i + L_i \tilde{\delta}_{\perp} k^i \right],$$

where we have used (22), the Leibniz rule, and we have introduced the quantity  $L_i = dL/dk^i = L^* \hat{k}_i$ . An asterisk denotes a derivative of  $L$  with respect to its argument,  $\kappa_1$ , and  $\hat{k}^i = k^i/k$ . We can now use (23) to obtain

$$\delta_\perp S = \int d\tau \left[ k_i L \Phi^i + L_i \tilde{D}^2 \Phi^i - L_i k^i k_j \Phi^j \right].$$

Integrating by parts, we reduce this expression to

$$\delta_\perp S = \int d\tau \left[ \tilde{D}^2 L_i + (L - L_j k^j) k_i \right] \Phi^i + \int d\tau \left[ L_i \tilde{D} \Phi^i - \Phi^i \tilde{D} L_i \right]'. \quad (41)$$

### 5.1. Equations of motion

By comparison of (41) with (39), we recognize by inspection the Euler-Lagrange derivative,  $E^{(1)}_i$  (the superscript refers to the first-curvature dependance), so that the equations of motion are

$$E^{(1)}_i = \tilde{D}^2 L_i + (L - L_j k^j) k_i = 0. \quad (42)$$

These are non-linear ordinary differential equations of fourth order in the embedding functions, or of second order in the curvatures. It is instructive to compare this expression with the considerably more complicated corresponding equations (at least prior to gauge fixing) written directly in terms of the embedding functions.

It is clear from the equations of motion that the normal forms introduced in (7)  $T_{i_1 \dots i_{N-r}}^{(N)} = 0$  on shell for all  $r > 2$ . This is because the equations of motion express  $\tilde{D}^2 L_i$  (and therefore  $\tilde{D}^2 k_i$ ) in terms of  $k_i$ . This implies that the third FS curvature  $\kappa_3$  will vanish. The classical dynamics is confined to at most a three dimensional Minkowski subspace. Note that if Minkowski space is replaced by an arbitrary background, a non-diagonal term proportional to the Riemann curvature of the background gets introduced into (42) which spoils this confinement.

An alternative way to see that  $\kappa_3$  vanishes is to use  $L_i = L^* \hat{k}_i$  in the equations of motion. We have  $L_i k^i = L^* \kappa_1$ , and the equations of motion assume the form

$$L^* \tilde{D}^2 \hat{k}^i + 2L^{*'} \tilde{D} \hat{k}^i + (L^{*''} + L - L^* \kappa_1) \hat{k}^i = 0.$$

We now project this equation into the three independent directions in the normal plane  $\eta_1, \eta_2, \eta_3$  and then we find that they imply, along  $\eta_3$ ,

$$E^{(1)}_3 = L^* \kappa_2 \kappa_3 = 0, \quad (43)$$

*i.e.* the vanishing of  $\kappa_3$ . Moreover, we obtain along  $\eta_2$ ,

$$E^{(1)}_2 = -2L^{*'} \kappa_2 - L^* \kappa_2' = 0. \quad (44)$$

This equation determines  $\kappa_2$  in terms of  $L^*$ , or equivalently in terms of  $\kappa_1$ . It can be easily integrated to give the relation

$$(L^*)^2 \kappa_2 = \text{const.} \quad (45)$$

As we will show below, this conservation law can be interpreted in terms of conservation of the spin. The remaining projection of the equations of motion can be written as

$$E^{(1)}_1 = L^{*''} + (L - L^* \kappa_1) \kappa_1 - L^* \kappa_2^2 = 0. \quad (46)$$

We see that the second FS curvature contributes to the “driving force” in the equations of motion. When  $\kappa_2$  is expressed via (45) in terms of  $\kappa_1$ , the dynamical problem is

reduced to the motion of a one-dimensional fictitious particle with  $\kappa_1$  as position variable.

A more systematic way to arrive at (43), (45), (46) is desirable. This is possible by choosing the independent normal variations to be those projected along the FS basis. Specifically, we have

$$\begin{aligned}\delta S &= \int d\tau \{L^* \delta_\perp \kappa_1 + L \kappa_1 \Psi^1\} + \int d\tau (L\Phi)' \\ &= \int d\tau [L^{*''} - (\kappa_1^2 + \kappa_2^2)L^* + L\kappa_1] \Psi_1 + \int d\tau [L^* \kappa_2' - (2\kappa_2 L^*)'] \Psi_2 \\ &\quad + \int d\tau L^* \kappa_2 \kappa_3 \Psi_3 + \int d\tau [L\Phi + 2\kappa_2 L^* \Psi_2]' ,\end{aligned}\tag{47}$$

using the expression (33) for  $\delta_\perp \kappa_1$  in the normal variation of the action, and (40) for the tangential variation of the action. We immediately read off the equations of motion in the form (43), (45), (46).

Let us consider some specific examples beginning with the degenerate ones. For the familiar massive relativistic particle,  $L = -m$ , so that  $L_i = 0$ , and the equations of motion reduce to  $m\kappa_1 = 0$ , *i.e.* the vanishing of the particle acceleration. The linear correction to the relativistic particle,  $L = -m + \alpha\kappa_1$ , where  $\alpha$  is constant is also special. Now (45) gives  $\kappa_2$  is a constant and (46) reads  $m\kappa_1 + \alpha\kappa_2^2 = 0$  so that  $\kappa_1$  is also. If  $m = 0$ , then  $\kappa_2 = 0$  and the motion is confined to a plane where the linear action is topological (with a vanishing Euler-Lagrange derivative almost everywhere) [3]. We will have more to say about this case later. In all other cases,  $L^*$  is not constant, and (46) is of higher order. An interesting non-polynomial model defined by  $L = -\sqrt{k_0^2 - \kappa_1^2}$ , with  $k_0 = \text{constant}$ , describes a trajectory with a bounded acceleration  $k_0$  [17] (see also [18] for a discussion of bounded acceleration). It has the remarkable feature of having equations of motion linear in the variable  $L_i$ . Indeed, since  $L_i = \kappa_i/L$ , they take the form,  $(\tilde{D}^2 - k_0^2)L^i = 0$ , which describes an upside down harmonic oscillator. Note that as  $k \rightarrow k_0$  we have that  $L^i \rightarrow \infty$ . It is interesting to observe that in the corresponding theory described perturbatively by  $L = -k_0 + (1/2k_0)\kappa_1^2 + \dots$ , the equations of motion are not harmonic at any finite order in the expansion.

It is well known that the equations of motion (42) are integrable;  $\kappa_1$  is determined as a quadrature. For a relativistic particle, this was shown by Nesterenko *et al.* using a FS basis [12]. In the next section we will offer an alternative proof which exploits the conserved charges associated with the Poincaré symmetry of these models. The advantage of this technique is that it can be exploited at higher orders to identify non-trivial first integrals of the equations of motion, a task which is not at all obvious generalizing the methods used after (43).

## 5.2. Invariants

The Noether charge  $Q$  for  $K(\kappa_1)$  models is given by

$$Q = L\Phi + L_i \tilde{D}\Phi^i - \Phi^i \tilde{D}L_i = L\Phi + L^* \Psi_1' - L^{*'} \Psi_1 + 2L^* \kappa_2 \Psi_2 .\tag{48}$$

The first expression is obtained by exploiting Eq.(40) and (41) and comparing with (39); for the second we use (47) instead of (41) ( $L^i = L^* \hat{k}^i$ ). From the Noether charge we obtain the conserved linear momentum by specializing the arbitrary deformation

to a constant infinitesimal translation,  $\delta X^\mu = \epsilon^\mu$ , and setting  $Q = \epsilon_\mu P^\mu$ . With the help of (5) or (2) this gives for the conserved linear momentum,

$$P^\mu = (L_i k^i - L) X^{\mu'} - \left( \tilde{D} L^i \right) n^\mu{}_i = (L^* \kappa_1 - L) X^{\mu'} - L^{*'} \eta_1^\mu + \kappa_2 L^* \eta_2^\mu. \quad (49)$$

Note that  $P^\mu$  in general possesses non-vanishing projections along the first two FS normal directions. For the relativistic free particle,  $L = -m$ , we recover the familiar expression  $P^\mu = m X^{\mu'}$ . The momentum is purely tangential to the worldline. As long as the worldline remains timelike the momentum will be also. In general, however, when  $L_i \neq 0$ , there will be a non-trivial normal component. It follows that even if the trajectory is timelike, the momentum  $P^\mu$  need not be. The invariant mass is

$$M^2 = -P^\mu P_\mu = (L_i k^i - L)^2 - \tilde{D} L_i \tilde{D} L^i = (L^* \kappa_1 - L)^2 - (L^{*'})^2 - (L^*)^2 \kappa_2^2. \quad (50)$$

The subtracted tachyonic terms lower the mass. They vanish if and only if  $\tilde{D} L^i = 0$ . These are the *static* solutions for this system. If  $(L - L_i k^i)^2 < \tilde{D} L_i \tilde{D} L^i$ ,  $P^\mu$  is spacelike. The occurrence of tachyonic solutions is a well-known feature of higher derivative relativistic theories. This expression makes explicit that the mass is lowered both by variations in the acceleration and by excursions away from the plane determined by its velocity and its acceleration. In the following we will assume that  $M^2 \geq 0$ .

In a similar way from the Noether charge we obtain the conserved angular momentum by specializing the arbitrary deformation to a spacetime infinitesimal Lorentz transformation,  $\delta X^\mu = \omega^\mu{}_\nu X^\nu$  with the constants  $\omega_{\mu\nu} = -\omega_{\nu\mu}$ , and setting  $Q = \omega_{\mu\nu} M^{\mu\nu}$ . Using the first two of the FS equations (2) this gives the conserved angular momentum

$$M^{\mu\nu} = P^{[\mu} X^{\nu]} + N^{\mu\nu}, \quad (51)$$

where

$$N^{\mu\nu} = L^i n^{[\mu}{}_i X^{\nu]'} = L^* \eta_1^{[\mu} X^{\nu]'} \quad (52)$$

Note that  $M^{\mu\nu}$  is not generally of the simple orbital form unless  $L_i = 0$ , *i.e.* for the special case of the standard relativistic particle with  $L = -m$ . The bivector  $N^{\mu\nu}$  is the origin of the spin of curvature-dependent particle models. However,  $N^{\mu\nu}$  is not of the most general form: at this order there is no purely normal term of the form  $m^{ij} n^{[\mu}{}_i n^{\nu]}{}_j$ . In order to evaluate the spin of a curvature-dependent particle we define the Pauli-Lubanski pseudo-tensor ( $N \geq 2$ ,  $M^2 > 0$ )

$$S_{\alpha_1 \dots \alpha_{N-2}} = \frac{1}{\sqrt{(N-2)!}} \frac{1}{\sqrt{M^2}} \varepsilon_{\alpha_1 \dots \alpha_{N-2} \mu \rho \sigma} P^\mu M^{\rho \sigma}. \quad (53)$$

The Pauli-Lubanski pseudo-tensor picks up the non-orbital part of the angular momentum, and it is identified as the spin. Since  $\kappa_3$  vanishes as a consequence of (43), we can, without loss of generality, set  $N = 2$  so that

$$S = \frac{1}{\sqrt{M^2}} \varepsilon_{\mu \rho \sigma} P^\mu N^{\rho \sigma} = \frac{1}{\sqrt{M^2}} \varepsilon_{\mu \rho \sigma} n^\mu{}_i n^\rho{}_j X^{\sigma'} \left( \tilde{D} L^i \right) L^j. \quad (54)$$

Note that the tangential projection of the linear momentum plays no role at this order, therefore the spin is purely normal. The normal Levi-Civita density can be exploited to rewrite the Pauli-Lubanski pseudo-tensor in a more geometrical form,

$$S = \frac{1}{\sqrt{M^2}} \varepsilon_{ij} L^i \tilde{D} L^j = \frac{L^{*2}}{\sqrt{M^2}} \varepsilon_{ij} \hat{k}^i \tilde{D} \hat{k}^j = \frac{L^{*2}}{\sqrt{M^2}} \bar{\kappa}_2. \quad (55)$$

A necessary and sufficient condition for the Lagrangian  $L(\kappa_1)$  to be associated with a spinless particle is the vanishing of the second FS curvature,  $\kappa_2$ . If we consider  $L^i$  as a position variable, the first expression in (55) for the Pauli-Lubanski pseudo-tensor makes explicit that it can be seen as the angular momentum associated with invariance under normal rotations. The Poincaré Casimir  $S^2$  is given by

$$M^2 S^2 = (\tilde{D}L^i) (\tilde{D}L_i) L^j L_j - (L_i \tilde{D}L^i)^2 = (L^*)^4 \kappa_2^2, \quad (56)$$

which should be compared with the equation of motion in the direction  $\eta_2$ , as given by (45).

### 5.3. Integrability

In this section, we exploit the conserved quantities  $M^2$  and  $S^2$ , derived in the previous section, to provide an elementary proof of the integrability of the model described by a Lagrangian of the form  $L(\kappa_1)$ . We have from the FS forms of (50) and (56)

$$\begin{aligned} M^2 &= (L^* \kappa_1 - L)^2 - (L^*)'^2 - (L^*)^2 \kappa_2^2 \\ M^2 S^2 &= (L^*)^4 \kappa_2^2. \end{aligned} \quad (57)$$

The later determines  $\kappa_2$  as a function of  $\kappa_1$  with constant of proportionality given by the two Casimirs,  $M^2$  and  $S^2$ . Substituting into the former determines  $\kappa_1$  as a quadrature,

$$M^2 = (L^* \kappa_1 - L)^2 - (L^*)'^2 - \frac{M^2 S^2}{(L^*)^2}. \quad (58)$$

The analogue with the motion of a fictitious particle located at  $L_i$  moving in a central potential should now be obvious. To be explicit, note that its velocity can be decomposed into its projections along  $L_i$  (and so  $k_i$ ) and orthogonal to it  $((L^*)^2 = L_i L^i)$ :

$$\tilde{D}L_i = DL^* \hat{L}_i + (\tilde{D}L_i)_\perp, \quad (59)$$

where  $\tilde{D}L_\perp^j = (\delta^{jk} - \hat{k}^j \hat{k}^k) \tilde{D}L_k$ . The right hand side of (56) can alternatively be expressed in the form  $(L^*)^2 \tilde{D}L_\perp^j \tilde{D}L_{j\perp}$ . We note that the “kinetic” term  $(\tilde{D}L^i)(\tilde{D}L_i)$  appearing in the mass formula (50) can correspondingly be decomposed into its radial and tangential parts as

$$(\tilde{D}L^i)(\tilde{D}L_i) = |L^*'|^2 + (\tilde{D}L_\perp^j)^2. \quad (60)$$

We now exploit ‘angular momentum’ conservation, (56) to eliminate the tangential component in favor of a centrifugal potential. We obtain (58). If we identify  $L^i$  with position in  $N$  dimensional Euclidean space, then (58) describes the radial motion of a fictitious particle with unit mass moving in a central potential,

$$V(L^*) = -(L^i k_i - L)^2, \quad (61)$$

where the right hand side should be seen as an implicit function of  $\kappa_1$ , with angular momentum  $(N-1)S^2 M^2$ , and energy  $-M^2$ . We note that the potential is negative everywhere. In the degenerate case,  $L = -m + \alpha \kappa_1$ , from (58) we obtain the mass-spin relation  $M^2 = m^2/(1+S^2/\alpha^2)$ . In all other cases the expression (58) can be integrated to give

$$\tau = \int \frac{dL^*}{[-V(L^*) - M^2 + M^2 S^2 / L^{*2}]^{1/2}}. \quad (62)$$

The simplest model is described by  $L = -m + \alpha\kappa_1^2$ . The potential takes the form  $V = -(\alpha\kappa_1^2 + m)^2$ . For the bounded acceleration model, we find  $V = -k_0^2(L^{*2} + 1)$ . The potential typically is negative. For a given real mass  $L^*$  is bounded from below by the centrifugal barrier provided by the spin. A trajectory bouncing off this barrier will have  $L^*$  increase monotonically to infinity. While this feature does translate into a divergence of  $\kappa_1$  for all  $\kappa_1^n$  models,  $n \geq 2$ , it does not always imply the divergence of  $\kappa_1$  as the bounded acceleration model clearly indicates.

#### 5.4. Non-flat background

It is also instructive to approach this problem from an alternative point of view which does not rely (explicitly at least) on the global Poincaré invariance of the theory. Let us consider the antisymmetric normal tensor  $j^{ij}$  defined by

$$j^{ij} = L^i \tilde{D}L^j - L^j \tilde{D}L^i. \quad (63)$$

This is the angular-momentum tensor associated with the configuration variable  $L^i$  on the normal plane. Note, however, that  $j^{ij}$  is not the angular momentum associated with the lagrangian, treated as a function of  $L_i$ . Its derivative gives

$$\tilde{D}j^{ij} = (L^j \tilde{D}^2 L^i - L^i \tilde{D}^2 L^j) = (L^l k_l^i - L^i k^j_j) (L^j k^i - L^i k^j), \quad (64)$$

where we have exploited the Euler-Lagrange equations in the form (42) to obtain the second equality. Since  $L^i \propto k^i$ , we obtain  $\tilde{D}j^{ij} = 0$ . Each component of this “angular momentum” is conserved.

We note that on a general spacetime background the Euler-Lagrange equations for the particle are modified by the addition of a curvature dependent term as

$$\tilde{D}^2 L^i - (L - L^j k_j) k^i + R_{\mu\nu\rho\sigma} X^{\mu'} n^{\nu'} X^{\rho'} n^{\sigma'} j^j = 0, \quad (65)$$

where  $R_{\mu\nu\rho\sigma}$  is the background Riemann tensor. In particular, if the spacetime geometry has constant curvature,  $R_{\mu\nu\rho\sigma} = c(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ , where  $c$  is a constant, the curvature correction appearing in (65) is proportional to  $L^i$ . The  $j^{ij}$  continue to be conserved. This observation explains the integrability of the first curvature particle models in spaces of constant curvature found in Ref. [19].

#### 5.5. Scale invariance

Let us suppose that the particle model defined by  $L(\kappa_1)$  is also invariant under a scale transformation,  $\delta X^\mu = \varepsilon X^\mu$ , with  $\varepsilon$  constant. It is simple to show from the expression (48) for the Noether charge that the corresponding constant of the motion is given by  $W = P^\mu X_\mu$ . When the equations of motion are satisfied, the conservation of  $W$ ,  $W' = 0$ , implies the vanishing of the tangential component of the momentum, the one-dimensional analogue of the vanishing of the trace of the stress-energy tensor,  $L - L_i k^i = L - L^* \kappa_1 = 0$ . The unique solution, up to a multiplicative constant, is the degenerate  $L = \kappa_1$ . Note that in the special case  $N = 1$ , for a plane curve, there also exists the scale invariant action given by the signed curvature  $\overline{\kappa_1}$ , which is also a topological invariant, the winding number.

For this model, the momentum is purely normal,  $P^\mu = -(\tilde{D}\hat{k}^i)n^\mu_i = \kappa_2 \eta^\mu_2$ . The Euler-Lagrange equations assume the simple form  $\tilde{D}^2 \hat{k}_i = 0$ , which implies  $\kappa_2 = 0$ , so that the momentum vanishes. Moreover the spin vanishes as well and the classical motion lies on a plane. But the Lagrangian  $L = \kappa_1$  on a plane is simply the winding number if  $\kappa_1$  is positive everywhere. In any case, the Euler-Lagrange equations are identically satisfied almost everywhere.

### 5.6. Adding a derivative

At the next order in derivatives the action might depend on  $\kappa_1'$  or  $\kappa_2$ . While at first sight, it might appear that these models possess a comparable level of complexity, we will see that this is not the case. Let us consider briefly a model depending on the first derivative of the first curvature,  $\kappa_1'$ , for simplicity, with  $L = L(\kappa_1')$ . Again, the only variations give non-vacuous Euler-Lagrange equations are those along  $\Psi_1, \Psi_2, \Psi_3$ . ( $L^{**} = dL/d\kappa_1'$ ):

$$L^{**'} \kappa_2^2 \kappa_3^2 = 0, \quad (66)$$

$$(L^{**'})^4 \kappa_2^2 = \text{const.}, \quad (67)$$

$$L^{***'} - (L - L^{**} \kappa_1' + L^{**'} \kappa_1) \kappa_1 - L^{**'} \kappa_2^2 = 0. \quad (68)$$

The structure of these equations is very similar to that obtained with  $\kappa_1$ . As before,  $\kappa_3 = 0$  so that the motion is confined to a three-dimensional subspace of Minkowski spacetime. The second equation is a conservation law, which expresses  $\kappa_2$  in terms of derivatives of  $\kappa_1$ . The main difference is in the first equation, which is now of fourth order in derivatives of  $\kappa_1$ . The invariant mass is given by

$$M^2 = (L^{**} \kappa_1' - L^{**'} \kappa_1 - L)^2 - (L^{***'})^2 - \kappa_2^2 (L^{**'})^2; \quad (69)$$

the second Casimir is given by  $M^2 S^2 = (L^{**'})^4 \kappa_2^2$  which permits  $\kappa_2$  to be eliminated from Eq.(69). The resulting expression does not, however, ever provide a quadrature. Much more interesting are models depending on  $\kappa_2$  to which we will now turn.

## 6. Second-curvature models

We extend now our considerations to a relativistic particle whose dynamics is determined by an action depending on  $\kappa_2$ ,  $L = L(\kappa_2)$ . The tangential variation of the action is given by (40). For its normal variation of the action we use (34) and the Leibniz rule. There are non-vanishing variations along five of the FS normal directions ( $L_* = dL/d\kappa_2$ )

$$\begin{aligned} \delta_\perp S|_1 = & \int d\tau \left[ 2\kappa_2 \left( \frac{L_*'}{\kappa_1} \right)' + \frac{\kappa_2'}{\kappa_1} L_*' + L\kappa_1 - L_* \frac{\kappa_2}{\kappa_1} (2\kappa_1^2 + \kappa_3^2) \right] \Psi_1 \\ & + \int d\tau \left[ 2 \frac{\kappa_2}{\kappa_1} L_* \Psi_1' + \frac{\kappa_2'}{\kappa_1} L_* \Psi_1 - 2 \frac{\kappa_2}{\kappa_1} L_*' \Psi_1 \right]' \end{aligned} \quad (70)$$

$$\begin{aligned} \delta_\perp S|_2 = & \int d\tau \left[ \left( \frac{L_*'}{\kappa_1} \right)'' - \frac{L_*'}{\kappa_1} (\kappa_2^2 + \kappa_3^2) - (L_* \kappa_1)' - 2 \left( \frac{L_* \kappa_3^2}{\kappa_1} \right)' + \frac{L_* \kappa_3}{\kappa_1} \kappa_3' \right] \Psi_2 \\ & + \int d\tau \left[ -\frac{L_*}{\kappa_1} \Psi_2'' + \frac{L_*'}{\kappa_1} \Psi_2' + \frac{L_*}{\kappa_1} (3\kappa_3^2 + \kappa_1^2 + \kappa_2^2) \Psi_2 - \left( \frac{L_*'}{\kappa_1} \right)' \Psi_2 \right]' \end{aligned} \quad (71)$$

$$\begin{aligned} \delta_\perp S|_3 = & \int d\tau \left[ -\left( \frac{L_* \kappa_3}{\kappa_1} \right)'' - 2 \left( \frac{L_*' \kappa_3}{\kappa_1} \right)' + L_*' \frac{\kappa_3'}{\kappa_1} + \frac{L_* \kappa_3}{\kappa_1} (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \right] \Psi_3 \\ & + \int d\tau \left[ -3 \frac{L_* \kappa_3}{\kappa_1} \Psi_3' + \left( \frac{L_* \kappa_3}{\kappa_1} \right)' \Psi_3 - \frac{L_* \kappa_3'}{\kappa_1} \Psi_3 + 2 \frac{L_*' \kappa_3}{\kappa_1} \Psi_3 \right]' \end{aligned} \quad (72)$$

$$\delta_\perp S|_4 = \int d\tau \left[ 3 \frac{L_*' \kappa_3 \kappa_4}{\kappa_1} + 2 L_* \left( \frac{\kappa_3 \kappa_4}{\kappa_1} \right)' - \frac{L_* \kappa_3 \kappa_4'}{\kappa_1} \right] \Psi_4 - \int d\tau \left[ 3 \frac{L_* \kappa_3 \kappa_4}{\kappa_1} \Psi_4 \right]' \quad (73)$$



$$\delta_\perp S|_5 = - \int d\tau \frac{L_* \kappa_3 \kappa_4 \kappa_5}{\kappa_1} \Psi_5 \quad (74)$$

The corresponding Euler-Lagrange derivatives can then be read off by discarding total derivatives: For  $\Psi_5$  we have

$$E^{(2)}_5 = - \frac{L_* \kappa_3 \kappa_4 \kappa_5}{\kappa_1} = 0. \quad (75)$$

This equation of motion sets  $\kappa_5 = 0$ . Motion will be restricted to (at most) a 5-dimensional subspace. For  $\Psi_4$ , we have

$$E^{(2)}_4 = 3 \frac{L_*'}{\kappa_1} \kappa_3 \kappa_4 + 2 L_* \kappa_4 \left( \frac{\kappa_3}{\kappa_1} \right)' + \frac{L_*}{\kappa_1} \kappa_3 \kappa_4' = 0. \quad (76)$$

This equation of motion can be easily integrated to give  $L_*^3 \kappa_3^2 \kappa_4 / \kappa_1^2 = \text{const.}$ , which determines  $\kappa_4$  in terms of the lower curvatures. Note that this is the analogue of (45) for first curvature models, although in this case it does not appear to be derivable from a conserved charge.

The remaining three Euler-Lagrange derivatives are

$$\begin{aligned} E^{(2)}_3 = & -3 L_*'' \frac{\kappa_3}{\kappa_1} - 4 L_*' \left( \frac{\kappa_3}{\kappa_1} \right)' + L_*' \frac{\kappa_3'}{\kappa_1} - L_* \left( \frac{\kappa_3}{\kappa_1} \right)'' \\ & + \frac{L_* \kappa_3}{\kappa_1} (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) = 0, \end{aligned} \quad (77)$$

$$E^{(2)}_2 = \left( \frac{L_*'}{\kappa_1} \right)'' - \left( \frac{L_*'}{\kappa_1} \right) (\kappa_2^2 + \kappa_3^2) - (L_* \kappa_1)' - 2 \left( \frac{L_* \kappa_3^2}{\kappa_1} \right)' + \left( \frac{L_* \kappa_3}{\kappa_1} \right) \kappa_3' = 0, \quad (78)$$

$$E^{(2)}_1 = 2 \kappa_2 \left( \frac{L_*'}{\kappa_1} \right)' + \frac{\kappa_2'}{\kappa_1} L_*' + L \kappa_1 - L_* \frac{\kappa_2}{\kappa_1} (2 \kappa_1^2 + \kappa_3^2) = 0. \quad (79)$$

Let us remark that these three higher-order non-linear coupled ODEs do not appear to be tractable in general.

The corresponding Noether charge is given by

$$\begin{aligned} Q = & L \Phi + 2 \frac{\kappa_2}{\kappa_1} L_* \Psi_1' + \frac{\kappa_2'}{\kappa_1} L_* \Psi_1 - 2 \frac{\kappa_2}{\kappa_1} L_*' \Psi_1 - \left( \frac{L_*}{\kappa_1} \right) \Psi_2'' \\ & + \frac{L_*'}{\kappa_1} \Psi_2' - \left( \frac{L_*'}{\kappa_1} \right)' \Psi_2 + \left( \frac{L_*}{\kappa_1} \right) (3 \kappa_3^2 + \kappa_1^2 + \kappa_2^2) \Psi_2 \\ & - 3 \frac{L_* \kappa_3}{\kappa_1} \Psi_3' + \left( \frac{L_* \kappa_3}{\kappa_1} \right)' \Psi_3 - \frac{L_* \kappa_3'}{\kappa_1} \Psi_3 + 2 \frac{L_*' \kappa_3}{\kappa_1} \Psi_3 - 3 \frac{L_* \kappa_3 \kappa_4}{\kappa_1} \Psi_4. \end{aligned} \quad (80)$$

This permits us to write down the conserved momentum as

$$\begin{aligned} P^\mu = & (\kappa_2 L_* - L) X^{\mu'} - \frac{\kappa_2}{\kappa_1} (L_*)' \eta_1^\mu + \left[ - \left( \frac{L_*'}{\kappa_1} \right)' + \frac{L_*}{\kappa_1} (\kappa_1^2 + \kappa_3^2) \right] \eta_2^\mu \\ & + \left[ \left( \frac{L_* \kappa_3}{\kappa_1} \right)' + \frac{L_*' \kappa_3}{\kappa_1} \right] \eta_3^\mu - \frac{L_* \kappa_3 \kappa_4}{\kappa_1} \eta_4^\mu, \end{aligned} \quad (81)$$

which now possesses components along the first four FS normals. Let us emphasize the similarity of the first two components, along  $X'$  and  $\eta_1$  with the conserved momentum for first curvature models, given by (49). The non-trivial invariant  $M^2 = -P^\mu P_\mu$  can

be written down immediately, but it is not particularly illuminating. The angular momentum is given by  $M^{\mu\nu} = P^{[\mu} X^{\nu]} + N^{\mu\nu}$  where the non-orbital part is given by

$$N^{\mu\nu} = \frac{L_*'}{\kappa_1} \eta^{[\mu}_2 X^{\nu]'} - \frac{L_* \kappa_3}{\kappa_1} \eta^{[\mu}_3 X^{\nu]'} + L_* \eta^{[\mu}_1 \eta^{\nu]}_2, \quad (82)$$

which allows one to obtain the spin pseudo-tensor for this class of models.

It is straightforward to combine these results with the one obtained in Sect. 5, to obtain equations of motion and conserved quantities for models of the form  $L = L(\kappa_1, \kappa_2)$ . One has only to be careful not to double count the contribution from the normal variation of the infinitesimal proper time and from the parallel variation of the action. The special case of models linear in the FS curvatures will be considered elsewhere.

### 6.1. Integrability in three dimensions

In general, as we have remarked, the system of equations determining the motion appears to be intractable. Let us specialize to a three-dimensional ambient space, so that  $\kappa_3 = \kappa_4 = \kappa_5 = 0$ . Moreover, we specialize to the case of an action quadratic in the second curvature,  $L = -m + (1/2)\alpha\kappa_2^2$ . The case of an arbitrary  $L$  can be treated along the same lines. Only the degenerate case linear in the second curvature needs special treatment (see below). Surprisingly, we find that this model is integrable. Let us show how this comes about. The two surviving Euler-Lagrange equations (78), (79) reduce to

$$\frac{E^{(2)}_2}{\alpha} = \left( \frac{\kappa_2'}{\kappa_1} \right)'' - \left( \frac{\kappa_2'}{\kappa_1} \right) \kappa_2^2 - (\kappa_2 \kappa_1)' = 0, \quad (83)$$

$$\frac{E^{(2)}_1}{\alpha} = 2\kappa_2 \left( \frac{\kappa_2'}{\kappa_1} \right)' + \frac{(\kappa_2')^2}{\kappa_1} - \frac{3}{2} \kappa_2^2 \kappa_1 - \frac{m}{\alpha} \kappa_1 = 0. \quad (84)$$

The momentum takes the form

$$\frac{P^\mu}{\alpha} = \left( \frac{m}{\alpha} + \frac{\kappa_2^2}{2} \right) X^{\mu'} - \frac{\kappa_2 \kappa_2'}{\kappa_1} \eta_1^\mu - \left[ \left( \frac{\kappa_2'}{\kappa_1} \right)' - \kappa_1 \kappa_2 \right] \eta^{\mu}_2, \quad (85)$$

so that the invariant mass is

$$-\frac{M^2}{\alpha^2} = \left[ \left( \frac{\kappa_2'}{\kappa_1} \right)' - \kappa_1 \kappa_2 \right]^2 - \left( \frac{m}{\alpha} + \frac{\kappa_2^2}{2} \right)^2 + \frac{\kappa_2^2 \kappa_2'^2}{\kappa_1^2}. \quad (86)$$

We assume that  $M^2 > 0$ . Now, from (82), (85), one obtains that the spin pseudo-scalar is

$$\sqrt{M^2} S = \alpha \kappa_2 \left[ m + \frac{\alpha}{2} \kappa_2^2 - \alpha \left( \frac{\kappa_2'}{\kappa_1} \right)^2 \right]. \quad (87)$$

This can be rewritten as

$$\alpha \kappa_2'^2 + \kappa_1^2 \left( \frac{\sqrt{M^2} S}{\alpha \kappa_2} - \frac{\alpha}{2} \kappa_2^2 - m \right) = 0. \quad (88)$$

We can now exploit (84), (86), to obtain

$$\kappa_1^2 = \frac{\alpha^2 \kappa_2^4 (4\kappa_2 \sqrt{M^2} S - 4M^2 + 4m^2 - \alpha^2 \kappa_2^4)}{(\sqrt{M^2} S - \alpha \kappa_2^3)^2}. \quad (89)$$

This expression can now be substituted into (88) to provide a quadrature for  $\kappa_2$ .

Note the equation of motion along  $\eta_1$  (79) can be interpreted in terms of conservation of the spin, in the sense that  $\sqrt{M^2}S' = -(\kappa_2'/\kappa_1)E^{(2)}_1$ . Recall that for the first curvature models it was the equation of motion along  $\eta_2$  which was related to spin conservation. It follows, unfortunately, that our result does not appear to extend to the more general case of models that depend on both curvatures,  $L = L(\kappa_1, \kappa_2)$ . Two special cases of interest are, however, integrable: (i)  $L(\kappa_1)$  with a linear term in  $\kappa_2$  added; (ii)  $L = L(\kappa_2)$  in three dimensions with a linear term in  $\kappa_1$  added.

### 6.2. Scale invariance

Invariant under scale transformations, implies as before the vanishing of the tangential component of the momentum,  $L - L_*\kappa_2 = 0$ . As we found earlier for first curvature models, the unique solution, up to a multiplicative constant, is the degenerate  $L = \kappa_2$ .

The equation of motion along  $\eta_1$ , (79) reduces to  $\kappa_2(\kappa_1^2 + \kappa_3^2)/\kappa_1 = 0$ , which implies  $\kappa_2 = 0$ . Using this in (78) gives  $\kappa_1 = \text{const}$ . The momentum is purely normal,  $P^\mu = \kappa_1\eta^\mu_2$  and  $M^2 = -\kappa_1^2$ . The motion is tachionic. Moreover the spin vanishes.

### 6.3. Kuznetsov-Plyushchay model

Although the generic case  $L = L(\kappa_1, \kappa_2)$  appears to be intractable, there is in addition to the two integrable cases mentioned at the end of (6.1), a large class of models in three dimensions which is also. These are the models defined by

$$L = -m - \kappa_1 f(x), \quad x = \kappa_1/\kappa_2, \quad (90)$$

where  $f$  is an arbitrary function of its argument. These models were introduced by Kuznetsov and Plyushchay in [5] who exploited a Hamiltonian approach.

The interest of this class of models lies in the fact that they have simple solutions and that it is easy to isolate the non-tachyonic solutions. It follows from the form (90) that we have  $L^* = -f - xL_*$ . It is a straightforward to show that the momentum takes the remarkably simple form  $P^\mu = m(X^{\mu'} - x\eta^\mu_2)$ , so that, in agreement with [5],  $M^2 = m^2(1 - x^2)$ . From this expression, one obtains non-tachyonic solutions by requiring  $x^2 \leq 1$ . We emphasize that this is independent of the function  $f$ . The spin also takes the simple form  $S = -mf(x)/\sqrt{M^2}$ , from which it follows that  $f^2(x) = S^2(1 - x^2)$ . The solutions of this model are given by constant  $x$ , as one can also verify directly from the equations of motion.

## 7. Conclusions

We have examined the simplest relativistic object, a point particle, described by a local geometrical action by developing a theory of deformations tailored to the FS basis adapted to the particle worldline, and exploiting the Noether charges associated with Poincaré invariance. In particular, we demonstrated explicitly how these techniques may be applied not only to the well known class of models depending on the first FS curvature of the worldline, but also we break new ground using them to reveal remarkable properties of models depending on the second FS curvature. We are able to show that the latter models are also integrable in three dimensions. This is particularly surprising when we consider that the Euler-Lagrange equations describing the dynamics are sixth order non-linear coupled ODEs. Our work suggests a novel

approach to the Hamiltonian analysis for these models, as well as a point of departure for examining higher order relativistic extended objects.

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